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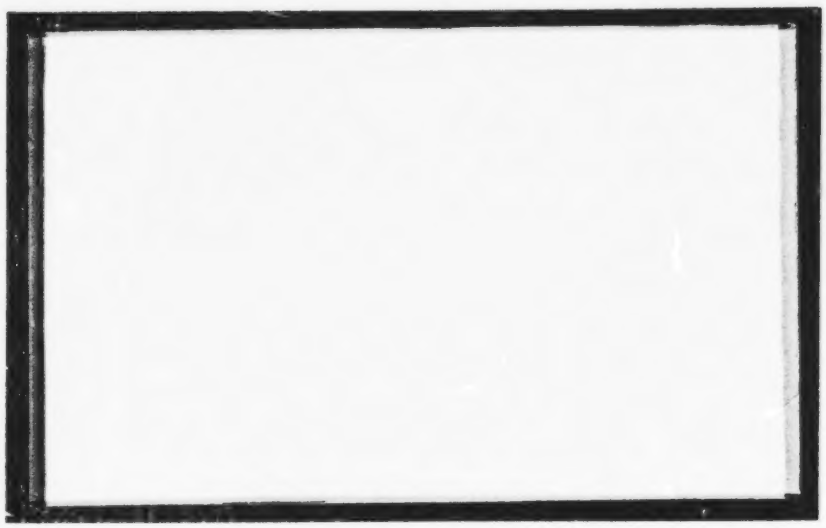

CENTRE DE RECHERCHE SUR LES TRANSPORTS

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ON COUNTING CONSTRAINED BINARY SEQUENCES

by

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ABSTRACT

Some common processes in Management Sciences and Engineering can be represented by sequences of 0 or 1 values, with constraints in the number of identical consecutive values in a sequence. For instance, in the case or urban traffic light control [6], a traffic light may be specified to stay in the same state, green or red, for at least 30 seconds and not more than 60 seconds. If a zero value in the sequence represents 5 seconds of red time and a 1 value, 5 seconds of green time, then the 0-1 sequence is required to have at least $L_1 = 6$ and not more than $L_2 = 12$ identical consecutive values in order to be admissible. Another illustration would be the case of a population with individuals with life duration of L_2 periods that start producing one offspring at the end of each period of their life, after having lived throug $L_1 - 1$ periods of "unproductive" maturation. Then the number of admissible sequences of n terms corresponds to the populaton size after n periods. Constrained binary sequences also occur in a variety of scheduling problems [7].

This paper gives a rigorous proof to the intuitive feeling that the number of admissible 0-1 sequences grows exponentially with the number of terms in a sequence. Consequences for optimal control with these sequences are then explored.

I EXPONENTIAL GROWTH OF CONSTRAINED 0-1 SEQUENCES

Theorem 1

The number of admissible 0-1 sequences of n terms, $n = 1,2,\ldots$ etc. is $o(c^n)$ as $n \to \infty$, where c is the unique real root between 1 and 2 for the polynomial equation,

$$x^{L_2} - \sum_{i=0}^{L_2-L_1} x^i = 0 \qquad (1)$$

To the author's knowledge, this result appears to be new and generalizes some earlier ones concerning Fibonacci sequences [4]. In order to establish the theorem, a few definitions and lemmas are first presented.

Definition 1

A sub-sequence comprising all consecutive identical values of a given 0-1 sequence is called a phase of that sequence. Phase length denotes the number of terms in a phase. The successive phase lengths in a sequence are listed as ph_1, ph_2, ... etc.

With each term u_k in a sequence:

$$\{u_k \mid k = 1,2,\ldots,n;\ u_k \in \{0,1\}\}$$

a number p_k is associated, called the phase state associated with term u_k; p_k is the maximum value p, such that

$$u_{k-i} = u_k \quad \text{for} \quad i = 1,\ldots, p - 1.$$

Index k associated with term u_k is also referred to as the stage index of the k^{th} stage.

Definition 2

A sequence $\{u_k\}_{k=1,\ldots,n}$ of values 0 or 1 is said to be admissible if at least L_1 ($L_1 \geq 1$) and not more than L_2 ($L_2 \geq L_1$) consecutive terms of that sequence have an identical value; however the following exceptions are conceded:

- The constraint of at least L_1 consecutive identical values is not applied to the last phase (on the right) of the sequence;

- As sequence $\{u_k\}_{k=1,\ldots,n}$ is considered to be a continuation of a sequence that terminated with value u_0 in phase state p_0. The first phase of the sequence is submitted to the special restrictions:

$$L_1 - p_0 \leq ph_1 \leq L_2 - p_0$$

and $u_1 = u_0$ if $p_0 < L_1$, where u_0 and p_0 are treated as initial conditions.

Let S_n denote the number of admissible sequences of n terms, $n = 1,2,\ldots$ etc, with initial conditions u_0, p_0.

Lemma 1

$0 < S_n \leq S_{n+1}$, $n = 1,2,\ldots$, and $S_n \to +\infty$ as $n \to +\infty$.

Proof:

Clearly $S_1 \geq 1$. For $n > 1$, any admissible sequence $\{u_k\}_{k=1,\ldots,n}$ can be extended by at least one value for u_{n+1}, thus $S_{n+1} \geq S_n$. Since after L_1 stages there will be at least one admissible sequence terminating with phase state L_1. The value of term u_{n+L_1+1} will be free to choose as 0 or 1, thus giving rise to one additional sequence; consequently $S_{n+L_1+1} \geq S_n + 1$. (In fact, Lemma 3 establishes that there exists a value m such that $S_{n+1} \geq S_n + 1$ for all $n > m$).

Lemma 2

The number S_n of admissible sequences obeys the recursion:

$$S_n = S_{n-L_1} + \ldots + S_{n-L_2} \tag{2}$$

Proof:

Let x_n^i denote the number of admissible sequences that are in phase state i at stage n, where i is any element from the set $\{1, \ldots, L_2\}$.

Let X_n be defined as $X_n = [x_n^1, \ldots, x_n^{L_2}]^T$ when T denotes the transpose operation.

Since S_n is the sum of the numbers of admissible sequences in all possible phase states,

$$S_n = C X_n \tag{3}$$

where C is a $1 \times L_2$ vector of ones.

The possible phase states of the n^{th} terms of an admissible sequence are $p_n = 1, \ldots, L_2$. Let sequence $\{u_k\}_{k=1,\ldots,n}$ be augmented by one more term on the right; the possible phase state transitions from stage n to stage $n + 1$ are represented in Figure 1, where an arrow denotes the possibility of a phase state transition.

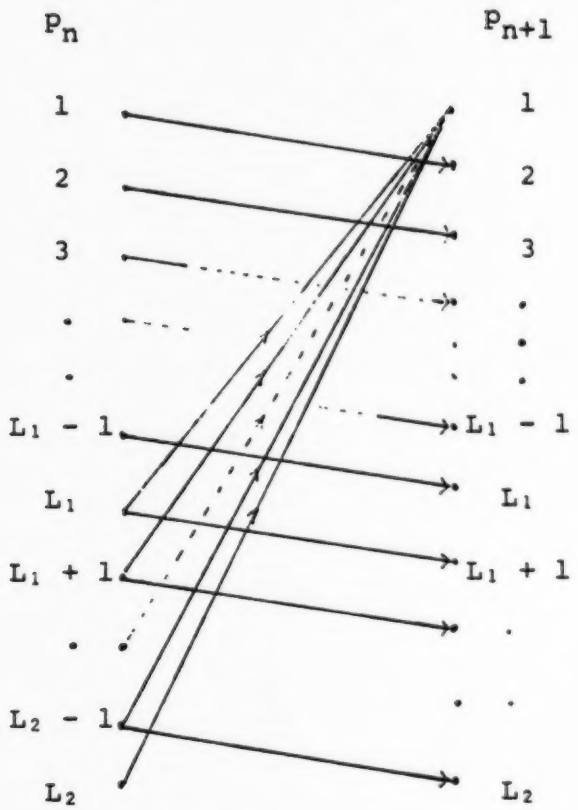

Fig. 1 - Phase state transitions.

The graph in Figure 1 is used to count the sequences of $n + 1$ terms terminating in the various possible phase states.

The number of sequences that switch in value from term n to term $n + 1$ is x_{n+1}^1, consequently:

$$x_{n+1}^1 = x_n^{L_1} + \ldots + x_n^{L_2} \tag{4}$$

and corresponding to sequences not switching in value,

$$x_{n+1}^j = x_n^{j-1}, \quad j \in \{2,\ldots,L_2\} \tag{5}$$

Equations (4) and (5) are now rewritten in matrix notations as,

$$X_{n+1} = A\, X_n \tag{6}$$

where A is a $L_2 \times L_2$ matrix of the form:

$A =$

states \ states	1	2	3	·	·	L_1	·	·	·	L_2
1	0	0	0	·	0	1	1	·	1	1
2	1	0	0	·	0	0	0	·	0	0
3	0	1	0	·	0	0	0	·	0	0
·	·	·	·	·	·	·	·	·	·	·
·	0	0	0	·	0	0	0	·	0	0
L_1	0	0	0	·	1	0		·	0	0
·	0	0	0	·	0	1	0	·	0	0
·	0	0	0	·	0	0	1	·	0	0
·	0	0	0	·	0	0	0	·	0	0
L_2	0	0	0	·	0	0	0	·	1	0

Suppose X_1 given arbitrary; then by (6)

$$X_n = A^{n-1}X_1 \qquad n = 1,2,\ldots, \text{ etc.}$$

(with $A^0 = I$, the identity matrix) and by (3)

$$S_n = C\,X_n = C\,A^{n-1}\,X_1.$$

The lemma will be established if it is true that

$$C\,A^{n-1}\,X_1 = C(A^{n-1-L_1} + \ldots + A^{n-1-L_2})X_1$$

and, a fortiori, if,

$$A^{n-1} = A^{n-1-L_1} + \ldots + A^{n-1-L_2}$$

or equivalently,

$$A^{L_2} = A^{L_2-L_1} + \ldots + A + I \tag{7}$$

Identity (7) can be established by considering the determinant equation:

$$|A - x\,I\,| = 0 \tag{8}$$

For the case $L_1 > 1$, the determinant in (8) is:

$$\begin{vmatrix} -x & 0 & \cdot & 1 & \cdot & 1 & 1 \\ 1 & -x & \cdot & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & -x & & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -x & 0 \\ 0 & 0 & 0 & 0 & \cdot & 1 & -x \end{vmatrix}$$

Expanding it with respect to the first row gives,

$$|A - x\,I| = (-x)^{L_2} + \sum_{i=L_1}^{L_2} (-1)^{i+1}\ \text{Det}\ (i) \qquad (9)$$

where Det (i) is the minor of matrix A corresponding to the i^{th} element of the first row of A - xI. More specifically:

$$\text{Det}\ (i) = \begin{vmatrix} M_1 & 0 \\ \hline 0 & M_2 \end{vmatrix}$$

where M_1 is a $(i - 1) \times (i - 1)$ matrix of the form:

$$M_1 = \begin{vmatrix} 1 & -x & 0 & 0 & 0 \\ 0 & 1 & -x & 0 & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

and M_2 is a $(L_2 - i) \times (L_2 - i)$ matrix of the form:

$$M_2 = \begin{vmatrix} -x & 0 & \cdot & 0 & 0 \\ 1 & -x & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & -x & 0 \\ 0 & 0 & \cdot & 1 & -x \end{vmatrix}$$

Clearly $|M_1| = 1$ and $|M_2| = (-x)^{L_2-i}$. It is a standard result of linear algebra [3] that

$$\text{Det}\ (i) = |M_1| \times |M_2|$$

so

$$\text{Det}\ (i) = (-x)^{L_2-i}$$

Consequently,

$$|A - x\,I| = (-x)^{L_2} + \sum_{i=L_1}^{L_2} (-1)^{i+1} (-x)^{L_2-i}$$

The characteristic equation of A is therefore:

$$(-1)^{L_2} x^{L_2} + \sum_{i=L_1}^{L_2} (-1)(-1)^{L_2} x^{L_2-i} = 0$$

i.e.

$$x^{L_2} - \sum_{i=L_1}^{L_2} x^{L_2-i} = 0 \qquad (10)$$

It is easy to verify that this equation also holds for the case $L_1 = 1$.

By Cayley-Hamilton theorem:

$$A^{L_2} = \sum_{i=L_1}^{L_2} A^{L_2-i}$$

which completes the proof.

The next step in the proof of Theorem 1 is to show that the characteristic polynomial equation, associated with difference equation (2), has a unique real root of maximum modulus.

Equation (10) is both the characteristic equation associated with the difference equation (2); and the characteristic polynomial equation for matrix A ; thus the roots of (10) are the eigenvalues of A . The eigenvalues of A therefore determine the behavior of the solutions of (2).

Lemma 3

Matrix A is primitive.

Proof:

It is known [5] that a matrix is primitive if and only if $A^m > 0$ (i.e. all components of A are positive) for some natural number m. It remains to show that $A^m > 0$ for some m.

Matrix A can be interpreted as the adjacency matrix [2] associated with the graph in Figure 1. Observe that any state is reachable from any initial state by a path composed of at most L_2 arcs. Let ℓ_{ij}, $\ell_{ij} > 0$, denote the length (in number of arcs) of a directed path connecting state i to state j. If m is a common multiple of the ℓ_{ij}'s, i = 1, ..., L_2; j = 1,...,L_2, then there is one directed path of length exactly m connecting each pair of states.

It is known [2] that the entries $a_{ij}^{(m)}$ of matrix A^m are equal to the number of directed paths connecting states i and j in exactly m arcs. Consequently $a_{ij}^{(m)} \geq 1 > 0$, and $A^m > 0$.

Lemma 4

Matrix A has a unique real positive root of maximum value.

The Perron-Frobenius theorem [1] is invoqued to prove that a primitive matrix has only one eigenvalue of maximum modulus with associated eigenvector having non negative components. Since by lemma 1, sequence $\{S_n\}_{n=1,2,\ldots}$ is non decreasing, this maximum modulus root must be a real number; otherwise the sequence would be oscillatory [4]. Finally, the non negativity of the eigenvector components insures that this root is positive.

It is a standard result of the theory of Difference Equations [4] that lemma 4 implies that the general solution of (2) is dominated by a term $k\,c^n$ where c denotes the maximum modulus root of A and k is a constant depending on the initial conditions selected for iterating (2).

It remains to show that $1 \leq c \leq 2$ and $k > 0$ in the case of the enumeration process for admissible 0-1 sequences.

If $L_1 = 1$ and $L_2 = +\infty$, then $S_n = 2^n$. On the other hand, if $L_2 = L_1$ then $S_n = 1^n = 1$. For the intermediate case $L_1 = 1,2,\ldots$; $L_2 > L_1$, one expects $1^n < S_n < 2^n$ and consequently $1 < c < 2$. A formal proof of this fact is obtained by rewriting (10) as, assuming $x \neq 1$ (i.e. $L_2 > L_1$),

$$\frac{1 - x^{-L_2}}{1 - x^{-L_1}} = x \qquad (11)$$

Let $f(x)$ denote the right-hand-side of (11). A simple exercise in calculus shows that:

$$\lim_{x \to 1} f(x) = \frac{L_2}{L_1} > 1$$

On the other hand, $f'(x) < 0$ for $x > 0$, and $f(2) < 2$; taking this into account as well as the continuity of $f(\cdot)$ over $(0,\infty)$ gives assurance that equation (11) has a unique real positive root which, for $L_2 > L_1$, is located in interval $(1,2)$. Finally, supposing $k \leq 0$ would result in $S_n \leq 0$ for some n; according to lemma 1, this cannot be the case, thus $k > 0$. This completes the proof of theorem 1.

II NUMERICAL ILLUSTRATIONS

Example 1

If $L_1 = 1$ and $L_2 = 2$ then (2) is the Fibonacci recursion [4] and

$$S_n \sim o\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n\right).$$

Example 2

If $L_1 = 5$ and $L_2 = 10$ then:

$$c = 1.\ 285,\ 199,\ 033,\ 245,\ \ldots$$

For $u_0 = 1$ and $p_0 = 3$, some computations give:

n	S_n	$k \sim S_n/c^n$	$S_n/2^n$
20	153	1.01224	$1.457\ 10^{-4}$
40	22,990	1.00629	$2.091\ 10^{-8}$
60	3,474,913	1.00628	$3.016\ 10^{-12}$

Thus the set of admissible sequences is a relatively small subset of the set of binary sequences.

To conclude this paper, a few results, closely linked with developments in section I, are briefly mentioned.

III ADDITIONAL RESULTS CONCERNING CONSTRAINED (0-1) SEQUENCES

Theorem 2

If the L_2 constraint on the maximum number of identical consecutive values for admissible 0-1 sequences is removed, then the number S'_n of admissible 0-1 sequences of n terms is such that $S'_n \sim o(d^n)$, as $n \to \infty$, with $1 < d \leq 2$.

The recurrence relation for S'_n can be shown to be:

$$S'_n = S'_{n-1} + S'_{n-L_1} \qquad (12)$$

(Observe that (2) can be written equivalently as

$$S_n = S_{n-1} + S_{n-L_1} - S_{n-1-L_2} \quad).$$

For the case $L_1 = 1$, recursion (12) takes the form $S'_n = 2^n S'_0$ and for the case $L_1 = 2$, the Fibonacci recursion is again obtained.

The next theorem provides some insight on an algorithm of type branch and bound applied in [6] for solving optimal control problems with control set in the form of admissible 0-1 sequences.

Theorem 3

For constrained 0-1 sequences with $1 \leq L_1 < L_2$,

$$\lim_{n \to +\infty} \frac{\sum_{k=1}^{n} S_k}{S_n} = \frac{c}{c - 1}$$

With theorem 1 established, the proof of this theorem is a standard exercise in analysis.

The algorithm in [6] is based on the fact that the set of admissible sequences can be structured as a tree, so that common subsequences are evaluated only once. Figure 2 is an illustration of such a tree for the case $L_1 = 2$ and $L_2 = 4$; a particular sequence is read from left to right, with a o corresponding to a value zero and • corresponding to a value one.

The computing effort associated with the algorithm in [6] is closely related to the number of edges in the tree corresponding to sequences of n terms. That number,

$$\sum_{k=1}^{n} S_k \quad ,$$

may be compared with the number $n\, S_n$ of edges in the

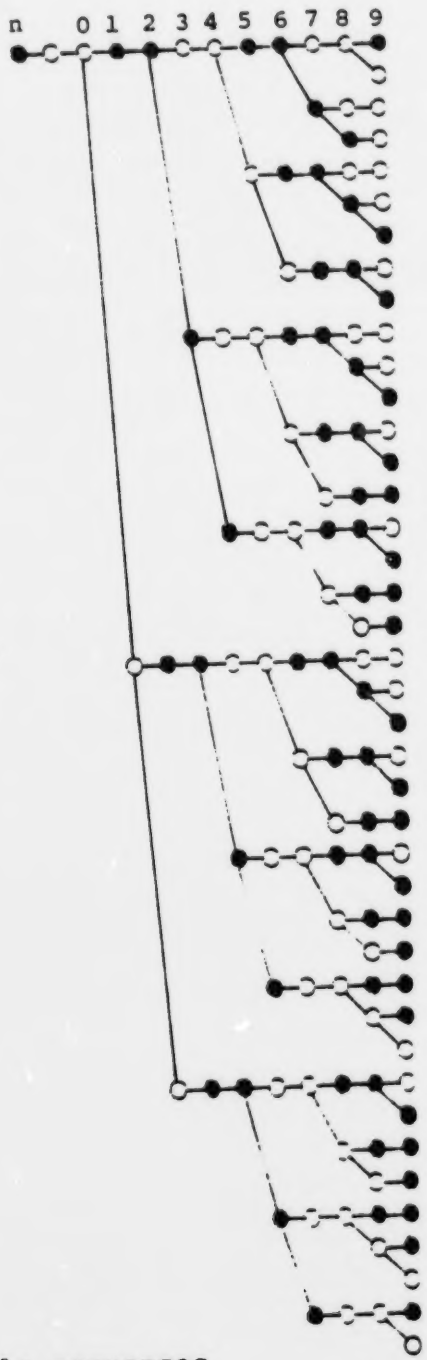

Fig. 2: Tree of 0-1 admissible sequences for $L_1 = 2$ and $L_2 = L_1$.

forest [2] of S_n trees measuring the computing effort associated with independent simulations of the S_n admissible sequences.

Theorem 2 indicates that use of the branch and bound approach divides by approximately $\frac{c}{n(c-1)}$ the computing effort required for identifying an optimal sequence of n terms. Since the number of admissible 0-1 sequences grows exponentially with n, there is a practical limit to the size of problems that can be solved by the algorithm in [6]. In the case of a traffic light control application, a few heuristics [6] combined with the branch and bound approach have significantly extended the size of the problems that can be solved practically.

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